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1992 J. Phys. A: Math. Gen. 25 6179

(http://iopscience.iop.org/0305-4470/25/23/015)

J. Phys. A: Math. Gen. 25 (1992) 6179–6186. Printed in the UK

NLRT formalism for asymmetric dichotomous Markov noise

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Received 18 June 1992

Abstract. We present a general theory of nonlinear relaxation times (NLRT) for stochastic transient dynamics of systems driven by an asymmetric dichotomous Markov noise (DMN). Two limiting cases of this general result are studied: the Poissonian white shot noise (WSN) and the Gaussian white noise (GWN).

1. Introduction

The theory of stochastic differential equations is a useful tool in the description of a great variety of non-equilibrium situations [1–6], mainly for the transient behaviour of the systems in which fluctuations play a very important role [7]. In particular, the theoretical study of transient relaxation of both unstable [8–16] and marginal [17–20] states has received increasing interest. The experimental analysis of the relaxation of unstable states has been devoted to the study of transient dynamics in lasers [21–23]. The mathematical modelling of these situations leads to a Langevin-type stochastic differential equation

$$\dot{x} = v(x) + g(x)\xi(t) \tag{1.1}$$

where $\xi(t)$ stands for a stochastic force or noise. In the study of the transient dynamics of the systems described by (1.1) three theoretical approaches have been proposed. The first one considers an evolution equation of a probability density, in which one asks for the time dependence of this density or its statistical moments. In general, this problem does not allow an exact analytical solution when applied to nonlinear systems, but standard approximation methods are available [2, 3]. The second one makes use of the first passage time (FPT) techniques, in which the random variable is the time necessary to cross a given boundary [1–3, 5]. The relevant quantities for using this method are the statistical moments of the FPT distribution. The third theoretical approach is the so-called nonlinear relaxation time (NLRT). Its definition is based on the transient evolution of the statistical moments and it parallels, to some extent, the FPT formulation.

The general formalism of the NLRT when applied to Gaussian white noise has been developed in [12, 13]. In the case of Gaussian coloured noise, an approximated solution of NLRT is given for small correlation times of the noise [14]. Both schemes have been used to study the decay of unstable states. The detection of weak optical signals is a recent application of this methodology [16]. Note should be taken that the formalism also admits a simple extension to non-Gaussian noises [24], two of which we are interested in.

Among those non-Gaussian noises on which we focus, the Poissonian white shot noise (WSN) and the dichotomous Markov noise (DMN) (also known as the random telegraph signal) are the most interesting ones which have been widely studied [24–28]. The purpose in this paper is to present the general theory of NLRT and its application to systems driven by DMN in the asymmetric case. We shall also show how the general result of the NLRT can be reduced in two limiting cases.

The main characteristic of the asymmetric DMN is that the random variable can take two values, Δ and Δ'. Each of these states has a given average duration τ_Δ and $\tau_{\Delta'}$ respectively, and then the transition from one state to another occurs at random time points. DMN is a simple example of a coloured noise, since its time correlation function is an exponentially decreasing function with finite-time correlation τ and intensity D

$$\langle \xi(t)\xi(t')\rangle = \frac{D}{\tau}e^{-|t-t'|/\tau}. \tag{1.2}$$

The shot noise is defined as the sum

$$\xi_{SN}(t) = \sum_i h(t-t_i) \tag{1.3}$$

where h is a given function and t_i are random time points distributed with a given average time spacing λ^{-1}. The probability to have n such time points in the time interval of duration t is then given by the Poisson distribution

$$P_n(t) = \frac{(\lambda t)^n}{n!}e^{-\lambda t}. \tag{1.4}$$

The Poissonian WSN is defined for the case in which h is proportional to a Dirac δ function,

$$\xi_{WSN}(t) = \sum_i w_i\delta(t-t_i) \tag{1.5}$$

where the δ pulses are weighted by w_i, which are random independent variables with a probability density $\phi(w)$. The process $\xi_{WSN}(t)$ is white but non-Gaussian, since its cumulants are all non-vanishing, although δ correlated [25, 26].

The natural definition of the NLRT associated with the average $\langle f(x(t))\rangle$ of a function of the relevant variable x is [12–14]

$$T = \int_0^\infty \frac{C(t)}{C(0)}\,dt \tag{1.6}$$

where $C(t) = \langle f(x(t))\rangle - \langle f(x)\rangle_{st}$, and $\langle f(x)\rangle_{st}$ and $\langle f\rangle_0 = \langle f(x(0))\rangle$ are the steady state and initial averages, respectively. The quantity defined in the integrand has no dimensions, so that the integral has the dimensions of time and characterizes the global relaxation of $f(x)$. Definition (1.6) gives us an exact solution of T through indirect methods which neither require explicit knowledge of $\langle f(x)\rangle$ nor $P(x,t)$ as functions of t.

The structure of this paper is as follows. In section 2, the general theory of NLRT for DMN, in the asymmetric case, is developed. In section 3, the NLRT for the WSN case is given [29]; this time scale, in turn, is shown to be a limiting case of the general result given in section 2. We show in section 4 that the NLRT for GWN is obtained as a particular case of the WSN. Finally the conclusions are given in section 5.

2. NLRT for asymmetric dichotomous Markov noise

We assume now that the noise $\xi(t)$ in equation (1.1) is a DMN, as was specified earlier. We denote by μ and μ' the respective transition probability per unit time between the two values of the noise. In the following we will always consider stochastic processes with vanishing average value. This implies that for DMN

$$\frac{\Delta}{\mu}+\frac{\Delta'}{\mu'}=0. \tag{2.1}$$

Then $\xi(t)$ is characterized by three independent parameters. The correlation function of the noise can be written as follows [3]

$$\langle\xi(t)\xi(t')\rangle=\frac{\mu\mu'}{(\mu+\mu')^2}(\Delta-\Delta')^2\exp[-(\mu+\mu')|t-t'|]. \tag{2.2}$$

It has a finite correlation time $\tau=(\mu+\mu')^{-1}=\Lambda^{-1}$, and the process defined by (1.1) is non-Markovian. However, it is possible to reduce the problem to a Markovian one. This Markovian formulation allows us to construct a set of coupled partial differential equations [27, 28]. Therefore the study of the NLRT can be made in terms of this formulation, following the procedures in [13, 24]. The idea of the method is based on the definition of a vector $\boldsymbol{P}(x,t)$ which obeys the evolution equation

$$\frac{\partial \boldsymbol{P}(x,t)}{\partial t}=\mathsf{L}(x)\boldsymbol{P}(x,t) \tag{2.3}$$

where

$$\boldsymbol{P}(x,t)=\begin{pmatrix}P(x,t)\\ \overline{P}(x,t)\end{pmatrix} \tag{2.4}$$

and the operator $\mathsf{L}(x)$ is the matrix

$$\mathsf{L}(x)=\begin{pmatrix}-\dfrac{\partial}{\partial x}\left[v(x)+\dfrac{\Delta+\Delta'}{2}g(x)\right] & \dfrac{\Delta-\Delta'}{2}\dfrac{\partial g(x)}{\partial x}\\ \mu-\mu'+\dfrac{\Delta-\Delta'}{2}\dfrac{\partial g(x)}{\partial x} & -\mu+\mu'+\dfrac{\partial}{\partial x}\left[v(x)+\dfrac{\Delta+\Delta'}{2}g(x)\right]\end{pmatrix}. \tag{2.5}$$

The meaning of the components in equation (2.4) is given in terms of the joint probability densities for the variables x and ξ, namely [6]

$$P(x,t)=P(x,\Delta;t)+P(x,\Delta';t) \tag{2.6a}$$

$$\overline{P}(x,t)=P(x,\Delta';t)-P(x,\Delta;t). \tag{2.6b}$$

The first component of equation (2.4) is the true probability density. Whereas the second component of (2.4) is not really a probability density but an auxiliary quantity.

In terms of this formalism, it is convenient to define some appropriate quantities that enable the NLRT (1.6) to be reduced to a quadrature. These quantities are

$$\boldsymbol{W}(x,t) = \boldsymbol{P}(x,t) - \boldsymbol{P}_{\mathrm{st}}(x) \tag{2.7}$$

and

$$\boldsymbol{\rho}(x) = \begin{pmatrix} \rho(x) \\ \overline{\rho}(x) \end{pmatrix} \tag{2.8}$$

where $\boldsymbol{\rho}(x)$ satisfies the equation

$$\boldsymbol{\rho}(x) = \int_0^\infty \boldsymbol{W}(x,t)\,\mathrm{d}t. \tag{2.9}$$

Therefore we find that the NLRT, defined in (1.6), can be written as

$$T = \frac{1}{C_0}\int_a^b f(x)\rho(x)\,\mathrm{d}x. \tag{2.10}$$

The calculation of equation (2.10) can be made after we find an expression for $\rho(x)$. First of all, equation (2.3) allows us to write the relationship between $\boldsymbol{W}(x,0)$ and $\boldsymbol{\rho}(x)$,

$$-\boldsymbol{W}(x,0) = \mathsf{L}(x)\boldsymbol{\rho}(x) \tag{2.11}$$

in such a way that the differential equation for $\rho(x)$ is given by

$$\rho'(x) - \left[\frac{\mathrm{d}}{\mathrm{d}x}\ln\left(\frac{g}{D_{\mathrm{eff}}(x)}\right) + \Lambda\left(\frac{v}{D_{\mathrm{eff}}(x)}\right)\right]\rho(x) = \frac{G(x)}{D_{\mathrm{eff}}(x)} \tag{2.12}$$

where the prime denotes the derivative with respect to x, and $D_{\mathrm{eff}}(x)$ is called the static effective diffusion [24], which reads as

$$D_{\mathrm{eff}}(x) = -[v(x) + \Delta g(x)][v(x) + \Delta' g(x)] \tag{2.13}$$

and

$$G(x) = \left[\Lambda + g\left(\frac{v}{g}\right)'\right]F(x) + \left[v + \frac{\Delta+\Delta'}{2}g\right]F'(x) + \frac{\Delta-\Delta'}{2}gh(x). \tag{2.14}$$

with

$$F(x) = \int_a^x \left[P_{\mathrm{st}}(x') - P_0(x')\right]\mathrm{d}x' \tag{2.15}$$

and

$$h(x) = \overline{P}_{st}(x) - \overline{P}_0(x). \tag{2.16}$$

The solution of the differential equation (2.12) is substituted into equation (2.10) and after an integration by parts, the NLRT for the stochastic systems derived by an asymmetric DMN will be

$$T = \frac{\Lambda}{C_0}\int_a^b \frac{I(x)}{D_{\text{eff}}(x)P_{st}(x)}\left\{\left[1+\Lambda^{-1}g\left(\frac{v}{g}\right)'\right]F(x) + \Lambda^{-1}\left[\left(v+\frac{\Delta+\Delta'}{2}g\right)\delta P - \frac{\Delta-\Delta'}{2}g\,\delta\overline{P}\right]\right\}\mathrm{d}x \tag{2.17}$$

where

$$I(x) = -\int_a^x \left(f(x') - \langle f\rangle_{st}\right)P_{st}(x')\,\mathrm{d}x' \tag{2.18}$$

$$P_{st}(x) \propto \frac{g(x)}{D_{\text{eff}}(x)}\exp\left(\Lambda\int^x \frac{v(x')}{D_{\text{eff}}(x')}\right)\mathrm{d}x' \tag{2.19}$$

and

$$\delta P = F'(x) = P_{st}(x) - P_0(x) \tag{2.20}$$

$$\delta\overline{P} = h(x) = \overline{P}_{st}(x) - \overline{P}_0(x) \tag{2.21}$$

where $\overline{P}_{st}(x)$ arises from a simple analysis of equation (2.3), then

$$\overline{P}_{st}(x) = \frac{2}{\Delta-\Delta'}\left[\frac{v}{g}+\frac{\Delta+\Delta'}{2}\right]P_{st}(x). \tag{2.22}$$

The structure of solution (2.17) is very similar to that found in [14] to first order for the correlation time τ for Gaussian coloured noise. In fact, we have a first contribution in (2.17) which would be a GWN solution of an effective Markovian problem. The second term is of order $\tau = \Lambda^{-1}$, and it is a term which couples information about the initial conditions of the system variable x and the noise variable, ξ. It also contains information about the type of model.

The initial decoupling between the system and noise variables, would be a simple hypothesis to describe the initial relaxation of the system. This assumption simplifies the second term of (2.17) to a simple quantity $\tau v(x)P_0(x)$. However, the case of great interest which connects with real situations, focuses on the distributed initial conditions and this means that the initial conditions have a certain probability density with finite width. The most general case would be the one in which the initial state corresponds to the steady state of a certain model with other parameters v^*, g^*, Δ^* and Δ'^*. This is $\overline{P}_0(x) = P^*_{st}(x)$, where P^*_{st} is the same as (2.22) but

$$\overline{P}^*_{st}(x) = \frac{2}{\Delta^*-\Delta'^*}\left[\frac{v^*}{g^*}+\frac{\Delta^*+\Delta'^*}{2}\right]P^{i}_{st}(x) \tag{2.23}$$

which reduces (2.17) to the following expression

$$T = \frac{\Lambda}{C_0}\int_a^b \frac{I(x)}{D_{\rm eff}(x)P_{\rm st}(x)}\left\{\left[1+\Lambda^{-1}g\left(\frac{v}{g}\right)'\right]F(x) \;-\Lambda^{-1}g\left[\left(\frac{v}{g}+\frac{\Delta+\Delta'}{2}\right)P_0(x)\right.\right.$$
$$\left.\left. +\left(\frac{\Delta-\Delta'}{\Delta^*-\Delta'^*}\right)\left(\frac{v^*}{g^*}+\frac{\Delta^*+\Delta'^*}{2}\right)\right]P^i_{\rm st}(x)\right\}\,{\rm d}x. \tag{2.24}$$

3. The NLRT for white shot noise (WSN)

In order to see the self-consistence of the general result obtained in section 2, we will show how equation (2.17) can be reduced, in an appropriate limit, to the NLRT for a system driven by a WSN.

For the WSN case we take the limits $\mu' \to \infty$, $\Delta' \to \infty$, with $\Delta'/\mu' = \omega_0 = cte$. The parameter μ plays the role of λ and Δ becomes equal to $-\lambda\omega_0$. In this case

$$\frac{1}{D_{\rm eff}(x)} = \frac{1}{\Lambda^{-1}D_{\rm eff}(x)} \to \frac{1}{D_{\rm s}(x)} \tag{3.1}$$

$$\frac{\Lambda^{-1}}{D_{\rm eff}(x)} = \frac{1}{D_{\rm eff}(x)} \to 0 \tag{3.2}$$

$$\frac{\Lambda^{-1}(\Delta+\Delta')}{D_{\rm eff}(x)} = \frac{\Delta+\Delta'}{D_{\rm eff}(x)} \to \frac{\omega_0}{D_{\rm s}(x)}. \tag{3.3}$$

In equation (2.17) we can note that the factor $-\Delta$ contributes to a constant term, and therefore it does not vanish. So we must analyse with some care all the second terms of (2.17). Let us define $J(x)$ as

$$J(x) = [v + \tfrac{1}{2}\Delta + \Delta' g]\delta P - \tfrac{1}{2}(\Delta - \Delta')g\;\delta\overline{P} \tag{3.4}$$

which can be written in an explicit way, taking into account equations (2.20)–(22) and (2.6), as

$$J(x) = 2[v + \tfrac{1}{2}(\Delta+\Delta')g]\delta P + vP_0(x) - g[\Delta P(x,\Delta,0) + \Delta' P(x,\Delta',0)]. \tag{3.5}$$

We now call

$$B(x) = \Delta P(x,\Delta,0) + \Delta' P(x,\Delta',0). \tag{3.6}$$

If we assume that at the time $t = 0$, the variable of the system x and the two values and of the noise are decoupled, and taking into account that both initial distributions are the same as the initial stationary probability (see [6], p 259), we obtain

$$B(x) = \frac{P_0(x)}{\Lambda}(\Delta\mu' - \Delta\mu') = 0. \tag{3.7}$$

Therefore, in the considered limits and with the additional hypothesis about the initial decoupling between the noise and system variables, the time scale for the WSN is given by [29]

$$T = \frac{1}{C_0} \int_a^b \frac{I(x)}{D_s(x) P_{st}(x)} [F(x) + \omega_0 g(P_{st}(x) - P_0(x))] \, dx \tag{3.8}$$

where $F(x)$ and $I(x)$ are the same as in (2.15) and (2.18) respectively; and

$$D_s(x) = \omega_0 g(\lambda - v)$$

$$P_{st}(x) \propto \frac{1}{(v - \lambda\omega_0 g)} \exp\left(- \int^x \frac{v(x')}{\omega_0 g(x')[v(') - \lambda\omega_0 g(x')]} \, dx' \right). \tag{3.9}$$

4. NLRT for Gaussian white noise (GWN)

This time scale can now be obtained from equation (3.8) in the limits $\omega_0 \to 0$, $\lambda \to \infty$, but $\lambda\omega_0^2 = D = cte$. In these conditions we get

$$\frac{1}{D_s(x)} \to \frac{1}{Dg^2} \tag{4.1}$$

$$\frac{\omega_0}{D_s(x)} \to 0. \tag{4.2}$$

Then, the NLRT for the dynamical system (1.1) driven by GWN reads

$$T = \frac{1}{C_0} \int_a^b \frac{I(x) F_1(x)}{Dg^2(x) P_{st}} \, dx \tag{4.3}$$

which is a known result and P_{st} corresponds to the expression for GWN [12–14].

5. Conclusions

We have obtained, in general circumstances, a formal and exact expression for the NLRT for characterizing the transient dynamics of systems driven by asymmetric dichotomous noise.

Expression (2.17) shows a quasi-Markovian contribution in the first term and the non-Markovian effects appear in the second term. The important point of our result is that, as in the coloured noise problem [14], the non-Markovian contribution shows a natural initial coupling between the system and noise variables. Therefore the NLRT is an appropriate quantity with which to study non-Markovian effects on initial conditions.

Acknowledgments

Financial support from Dirección General de Investigación Científica y Técnica (Projet PB90-0030, Spain) and CONACYT (México)-CSIC (Spain) are acknowledged.

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